Limits to the scope of applicability of *extended formulations* theory for LP models of combinatorial optimisation problems

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Abstract: The purpose of this paper is to bring to attention and to make a contribution to the issue of defining/clarifying the scope of applicability of *extended formulations* (EFs) theory. Specifically, we show that EFs theory is not valid for relating the sizes of descriptions of polytopes when the sets of the descriptive variables for those polytopes are disjoint, and that new definitions of the notion of 'projection' upon which some of the recent *extended formulations* works [such as Kaibel (2011), Fiorini et al. (2011, 2012a, 2012b, 2013), Faenza et al. (2012), Gillis and Glineur (2012) and Kaibel and Walter (2014), for example] have been based can cause those works to over-reach in their conclusions.

Keywords: extended formulations; combinatorial optimisation; computational complexity; linear programming.

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1 Introduction

The big challenge in combinatorial optimisation is that the sizes of natural/standard formulations for many of the most important problems grow exponentially with the number of objects about which the decisions must be made. Since the early days of the field of mathematical programming, it has been known that the issues of model size growth and of the ease-of-solution of models can often be improved by adding extra variables and constraints to a model, or, in other words, by ‘lifting’ a model into a higher-dimensional space. The study of the relationships between a given polytope and its ‘lifted’ version and the development of procedures for achieving ‘good’ ‘lifted’ polytopes are the subjects of *extended formulations* (EFs) theory. Hence, EFs theory is predicated on the notion of ‘projection’ by necessity.

A seminal development in EFs theory is that of Yannakakis (1991). This development was undertaken in the process of the assessment of the validity of a linear programming (LP) model for the travelling salesman problem (TSP) which had been proposed by Swart (1986, 1987). One of Yannakakis’ main results was the showing that a LP model which *projects* (see Section 2 for formal definitions) to the standard TSP polytope (see Lawler et al., 1985) cannot be polynomial-sized (in the number of cities), *if* it is symmetric. Although we have not had access to Swart’s model, the consensus of the research communities has been that its validity was refuted with this Yannakakis development. Hence, starting with Yannakakis (1991), EFs theory has been the single-most important paradigm for deciding the validity of proposed LP models for NP-Complete problems.

Over the past decade, perhaps due (at least in part) to the renewed interest in LP modelling of hard combinatorial optimisation problems (COPs) [see Diaby (2007; 2010a, 2010b, 2010c), and Diaby and Karwan (2015), in particular], there has been a renewed interest in EFs [see Conforti et al. (2010, 2013) and Vanderbeck and Wolsey (2010) for extensive reviews]. Recent EFs developments can be classified into three broad groups:

1 work focused on techniques for solving special-structured EFs

2 work aiming to propose ‘good’ EFs or procedures to obtain such

3 work aimed at establishing bounds on the minimum sizes of EFs for well-known polytopes.

An example of work in the first group is Sadykov and Vanderbeck (2013). Work in the second group include Gouveia et al. (2013), Ballerstein and Michaels (2014), Bärmann et al. (2014), Buchanan and Butenko (2014), Godinho et al. (2014), Lancia and Serafini (2014) and Leggieri et al. (2014). Examples of work in the third group include Kaibel (2011), Fiorini et al. (2011, 2012a, 2012b, 2013), Faenza et al. (2012), Gillis and Glineur (2012) and Kaibel and Walter (2014).

The focus of this paper is on developments aimed at establishing lower bounds on the sizes of EFs (i.e., work in the third group above). Because the issue of the scope of applicability of EFs theory in general has been a largely overlooked issue, some of the recent lower-bounding developments can lead to over-reaching claims (implied or explicitly-stated) about what is/is not possible in terms of LP modelling of combinatorial optimisation problems. The purpose of this paper is to make a contribution towards addressing this issue. Specifically, we will show that EFs theory is not valid for relating the sizes of descriptions of polytopes when the sets of the descriptive variables for those polytopes are disjoint, and that new definitions of the notion of 'projection' upon which some of the recent EFs works dealing with combinatorial optimisation problems [such as Kaibel (2011), Fiorini et al. (2011, 2012a, 2013), Faenza et al. (2012) and Kaibel and Walter (2014), for example] have been based can cause those works to over-reach in their conclusions when the sets of the descriptive variables for the polytopes being related are disjoint or can be made so after redundant variables and constraints (with respect to the optimisation problem at hand) are removed.

Kaibel and Weltge (2015) introduce a new paradigm, which is that of the 'relaxation complexity' of a polytope. The idea of this new paradigm is to try to establish lower bounds on the sizes of descriptions of relaxations (instead of extensions) of a polytope. They show that every relaxation of the standard TSP polytope which has the integrality property and no integrality gap (see Geoffrion, 1974) must have an exponential number of constraints. They also show that this is true for other hard combinatorial optimisation problems as well. However, the application of their results to models other than the natural formulations of COPs still requires the use of the notion of 'projection,' the same way as in EFs theory. Hence, our developments in this paper are applicable to Kaibel and Weltge (2015) as well, in the same way as they apply to EFs work in general.

It should be noted that our intent in this paper is not to claim the correctness or incorrectness of any particular model that may have been developed in trying to address the '$P = NP$' question. Our aim is, strictly, to bring attention to limits to the scope within which EFs theory is applicable when attempting to derive bounds on the sizes of linear programming models for combinatorial optimisation problems. In other words, the developments in the paper are not about deciding the correctness/incorrectness of any given LP model, but only about the issue of when such a decision (of correctness/incorrectness) is beyond the scope of EFs theory.

One of the most fundamental assumptions in EFs theory is that the addition of redundant variables and constraints to a given model of an optimisation problem at hand does not change the EFs relationships for that model. The key point of this paper is to show the flaw in this assumption, which is that it leads to ambiguity and degeneracy/loss of meaningfulness of the notion of EF when the sets of the descriptive variables for the polytopes involved are disjoint. We show that if redundant variables and constraints can be arbitrarily added to the description of a given polytope for the purpose of establishing EFs relationships, then every given mathematical programming model would be an EF of every other mathematical programming model, provided their sets of descriptive variables are disjoint, which would clearly mean a loss of meaningfulness of the notion (of EF).

Our developments in this paper were initially motivated by our realisation that the 'new' definition of EFs (Definition 2) which was first proposed in Kaibel (2011) and then subsequently used in other developments [such as Fiorini et al. (2011, 2012a) for example] becomes inconsistent with other and previous/'standard' definitions of EFs (see Definition 1) when the sets of the descriptive variables of the polytopes being related are disjoint. Comments we received in private communications and also in anonymous reviews on an earlier version of this paper were that our finding of inconsistency was 'obvious,' and that all of our developments were 'obvious' because of that. Hence, we believe it may be useful to recall at this point that the Fiorini et al. (2011, 2012a) work which is based on the 'new' definition in question has been highly-recognised, having received numerous accolades and awards. Also, more generally, and perhaps more deeply, Martin's formulation of the minimum spanning tree problem (MSTP; see Section 4.2 of this paper) is cited in almost every EFs paper in the current literature as a 'normal' EF of Edmonds' formulation of the MSTP (although with the acknowledgement that it 'escapes' the results for EFs of NP-Complete problems somehow). We argue that these facts highlight the need to bring our notions in this paper to the attention of the Optimisation communities in general, and of the EFs communities in particular.

The plan of the paper is as follows. First, we will review the background definitions in Section 2. Our main result (i.e., the non-validity/non-applicability of EFs theory when the sets of descriptive variables are disjoint) is developed in Section 3. In Section 4, we illustrate the discussions of Section 3 using the Fiorini et al. (2011, 2012a) developments, as well as Martin's (1991) LP formulation of the MSTP. In Section 5, we provide insights into the (correct) meaning/implication of the existence of a linear map between solutions of models, with respect to the task of solving an optimisation problem, when the set of descriptive variables of the models are disjoint. Finally, we offer some concluding remarks in Section 6.

The general notation we will use is as follows.

Notation

1 $\mathbb{R}$: set of real numbers

2 $\mathbb{R}_{\not<}$: set of non-negative real numbers

3 $\mathbb{N}$: set of natural numbers

4 $\mathbb{N}_{+}$: set of positive natural numbers

5 '$\mathbf{0}$': column vector that has every entry equal to 0

6 '$\mathbf{1}$': column vector that has every entry equal to 1

7 $(\cdot)^T$: transpose of $(\cdot)$

8 $Conv(\cdot)$: convex hull of $(\cdot)$.

2 Background definitions

For the purpose of making the paper as self-contained as possible, we review the basic definitions of an EF in this section.

Definition 1 ['Standard EF Definition' (Yannakakis, 1991; Conforti et al., 2010, 2013)]: An EF for a polytope $X \subseteq \mathbb{R}^p$ is a polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q}: Gx + Hw \leq g\}$ the projection, $\varphi_x(U): = \{x \in \mathbb{R}^p: (\exists w \in \mathbb{R}^q: (x, w) \in U)\}$, of which onto x-space is equal to X (where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).

Definition 2 ['Alternate EF Definition 1' (Kaibel, 2011; Fiorini et al., 2011, 2012a)]: A polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q}: Gx + Hw \leq g\}$ is an EF of a polytope $X \subseteq \mathbb{R}^p$ if there exists a linear map π: $\mathbb{R}^{p+q} \to \mathbb{R}^p$ such that X is the image of U under π [i.e., $X = \pi(U)$; where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$] [Kaibel (2011), Kaibel and Walter (2014) and Kaibel and Weltge (2015) refer to π as a 'projection'].

Definition 3 ['Alternate EF Definition 2' (Fiorini et al., 2012a)]: An EF of a polytope $X \subseteq \mathbb{R}^p$ is a linear system $U = \{(x, w) \in \mathbb{R}^{p+q}: Gx + Hw \leq g\}$ such that $x \in X$ if and only if there exists $w \in \mathbb{R}^q$ such that $(x, w) \in U$. (In other words, U is an EF of X if $(x \in X \Leftrightarrow (\exists w \in \mathbb{R}^q: (x, w) \in U)))$ (where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).

The purpose of this paper is to point out that the scope of applicability of EFs work based on the above definitions is limited to cases in which U cannot be equivalently reformulated (with respect to the task of optimising linear functions) in terms of the w variables only. For simplicity of exposition, without loss of generality, we will say that $G = \mathbf{0}$ in the above definitions if there exists a description of U which is in terms of the w-variables only and has the same or smaller complexity order of size. Or, equivalently, without loss of generality, we will say that $G \neq \mathbf{0}$ in the above definitions iff the x- and w-variables are required in every valid inequality description of U which has the same or smaller complexity order of size as the description at hand.

In particular, if every constraint of U which involves the x-variables is redundant in the description of U, then clearly, every one of those constraints as well as the x-variables themselves can be dropped (without loss, with respect to the task of optimising linear functions) from the description of U, with the result that U would be stated in terms of the w-variables only. Also, in some cases (all of) the constraints involving the x-variables may become redundant only after other constraints in the description of U are re-written and/or new constraints are added (as exemplified by the case of the MSTP in Section 4.2 of this paper). If either of these two cases is applicable, we will say that $G = \mathbf{0}$ in the above definitions. Otherwise, we will say that $G \neq \mathbf{0}$.

Remark 1: The following observations are in order with respect to Definitions 1, 2, and 3:

1 The statement of U in terms of inequality constraints only does not cause any loss of generality, since each equality constraint can be replaced by a pair of inequality constraints.

2 The system of linear equations which specify π in Definition 2 must be *valid* constraints for X and U. Hence, X and U can be respectively *extended* by adding those constraints to them, when trying to relate X and U using Definition 2. In that sense, Definition 2 'extends' Definitions 1 and 3.

3 All three definitions are equivalent when $G \neq \mathbf{0}$. However, this is not true when $G = \mathbf{0}$, as we will show in Section 3 of this paper.

4 In the remainder of this paper, we will use the term 'polytope' to refer to the polytope induced by a set of linear inequality constraints or the set of constraints itself, if this is convenient and does not cause ambiguity. □

3 Non-applicability and degeneracy conditions for EFs theory

Our main result will now be developed.

Theorem 1: EFs developments are not valid for relating the inequality descriptions of U and X in Definitions 1 to 3 when $G = \mathbf{0}$ in those definitions.

Proof: The proof will be in three parts. In Part 1, we will show that when $G = \mathbf{0}$, U cannot be an EF of X according to Definition 1. In Part 2, we will show that when $G = \mathbf{0}$, U cannot be an EF of X according to Definition 3. In Part 3, we will show that when $G = \mathbf{0}$, the EF notion under Definition 2 results in the condition that every given polytope is an EF of every other given polytope, provided their sets of descriptive variables are disjoint, which means that the (EF) notion becomes degenerate/meaningless. In the discussion, we will only consider the case in which $U \neq \varnothing$ and $X \neq \varnothing$, since the theorem is trivial when $U = \varnothing$ or $X = \varnothing$.

1 Consider Definition 1. Assume $G = \mathbf{0}$. Then, we have:

$$\begin{aligned}\varphi_x(U) &= \left\{x \in \mathbb{R}^p : \left(\exists w \in \mathbb{R}^q : (x, w) \in U\right)\right\}\\ &= \left\{x \in \mathbb{R}^p : \left(\exists w \in \mathbb{R}^q : Hw \leq g\right)\right\}\\ &= \mathbb{R}^p\\ &\neq X (\text{since } X \text{ is a polytope and thus bounded, whereas } \mathbb{R}^p \text{ is unbounded}).\end{aligned}$$

Hence when $G = \mathbf{0}$, U cannot be an EF of X according to Definition 1.

2 Consider Definition 3. Assume $G = \mathbf{0}$. Then, we have:

$$\begin{aligned}&\left(\exists w \in \mathbb{R}^q : \mathbf{0}x + Hw \leq g\right) \Leftrightarrow \left(\exists w \in \mathbb{R}^q : Hw \leq g\right)\\ &\Rightarrow \forall x \in \mathbb{R}^p, \exists w \in \mathbb{R}^q : (x, w) \in U.\end{aligned} \tag{1}$$

(1) implies:

$$\forall x \in \mathbb{R}^q \setminus X, \exists w \in \mathbb{R}^q : (x, w) \in U \ [\text{since } \mathbb{R}^p \supset X, \text{ and } \mathbb{R}^q \setminus X \neq \varnothing]. \tag{2}$$

(1) and (2) imply:

$$\left(\exists w \in \mathbb{R}^q : \mathbf{0}x + Hw \leq g\right) \not\Rightarrow x \in X. \tag{3}$$

From (3), the 'if and only if' stipulation of Definition 3 cannot hold in general. Hence, when $G = \mathbf{0}$, U cannot be an EF of X according to Definition 3.

3 Now, consider Definition 2. We will show that the EF notion under this definition becomes degenerate/meaningless when $G = \mathbf{0}$. The reasons for this are that a

polytope can also be stated in terms of its extreme points [see Rockafellar (1997, pp.153–172), among others], and that a linear map (as stipulated in the definition) could be inferred from this statement without reference to an inequality description of the polytope. The proof consists of a counterexample to the sufficiency of the existence of a linear map, as stipulated in the definition, for implying EFs relationships, as stipulated in the definition. Note that if $G = \mathbf{0}$ in Definition 2, then the linear inequality description of U involves the w-variables only.

For the sake of simplicity (but without loss of generality), let $\bar{U} \subset \mathbb{R}^5$ be described in the w-variables only as

$$\bar{U} = \left\{ w \in \mathbb{R}^5_{\not<} : w_1 + w_2 = 5;\ w_1 - w_2 = 1;\ w_3 + w_4 + w_5 = 0 \right\}. \tag{4}$$

Then, the vertex-description of $\bar{U}$ is

$$\bar{U} = \left\{ w \in \mathbb{R}^5_{\not<} : w \in Conv\left(\{(3, 2, 0, 0, 0)^T\} \right) \right\}.$$

Now, let $X \subset \mathbb{R}^3_{\not<}$ be specified by its vertex-description as

$$X = \left\{ x \in \mathbb{R}^3_{\not<} : x \in Conv\left(\{(2, 1, 5)^T\} \right) \right\}.$$

(In other words, X consists of the point in $\mathbb{R}^3$, $(2, 1, 5)^T$.)

Then, the following are true:

- $\left((x \in X) \text{ and } (w \in \bar{U}) \right) \Rightarrow x = Aw$,

  where, among other possibilities,

  $$A = \begin{bmatrix} -1 & 2.5 & 2 & 3 & 4 \\ 1 & -1 & 5 & 6 & 7 \\ -1 & 4 & 8 & 9 & 10 \end{bmatrix}. \tag{5}$$

  Hence, under Definition 2, $\bar{U}$ is an EF of every one of the infinitely-many possible inequality descriptions of X (since $x = Aw$ in the above is a linear map between $\bar{U}$ and X).

- Similarly,

  $\left((x \in X) \text{ and } (w \in \bar{U}) \right) \Rightarrow w = Bx$,

  where, among other possibilities,

  $$B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 4 & 0 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -11 & 1 \end{bmatrix}. \tag{6}$$

Hence, under Definition 2, every one of the infinitely-many possible inequality descriptions of X is an EF of the inequality description of $\bar{U}$ as stated in (4) (and, in fact, of everyone of the infinitely-many possible inequality descriptions of $\bar{U}$).

- Clearly, the EFs relations based on (5) and (6) above are degenerate/meaningless, since no meaningful inferences can be made from them in attempting to compare inequality descriptions of $\bar{U}$ and X. ■

A fundamental notion in EFs theory is that the addition of redundant variables and constraints to the inequality description of a polytope does not change the EFs relationships for that polytope. We use this fact to generalise the degeneracy/loss of meaningfulness which arises out of Definition 2 when $G = \mathbf{0}$ to Definitions 1 and 3, as follows.

Theorem 2: Provided redundant constraints and variables can be arbitrarily added to the descriptions of polytopes for the purpose of establishing EFs relationships under Definitions 1 to 3, the descriptions of any two given non-empty polytopes expressed in disjoint variable spaces can be respectively *augmented* into being EFs of each other. In other words, let $x^1 \in \mathbb{R}^{n_1} \, (n_1 \in \mathbb{N}_+)$ and $x^2 \in \mathbb{R}^{n_2} \, (n_2 \in \mathbb{N}_+)$ be vectors of variables with no components in common. Then, provided redundant constraints and variables can be arbitrarily added to the descriptions of polytopes for the purpose of establishing EFs relationships, the inequality-description of every non-empty polytope in x^1 can be *augmented* into an EF of the inequality-description of every other non-empty polytope in x^2, and vice versa.

Proof: The proof is essentially by construction.

Let P_1 and P_2 be polytopes specified as:

$$P_1 = \{x^1 \in \mathbb{R}^{n_1} : A_1 x^1 \le a_1\} \ne \varnothing \ \text{ (where } A_1 \in \mathbb{R}^{p_1 \times n_1}\text{, and } a_1 \in \mathbb{R}^{p_1}\text{);}$$

$$P_2 = \{x^2 \in \mathbb{R}^{n_2} : A_2 x^2 \le a_2\} \ne \varnothing \ \text{ (where } A_2 \in \mathbb{R}^{p_2 \times n_2}\text{, and } a_2 \in \mathbb{R}^{p_2}\text{).}$$

Clearly, $\forall (x^1,\, x^2) \in P_1 \times P_2, \forall q \in \mathbb{N}_+, \forall B_1 \in \mathbb{R}^{q \times n_1} : B_1 \ne \mathbf{0}, \forall B_2 \in \mathbb{R}^{q \times n_2} : B_2 \ne \mathbf{0},$ there exists $u \in \mathbb{R}^q_{\not<}$ such that the constraints

$$B_1 x^1 + B_2 x^2 - u \le 0 \tag{7}$$

are *valid* for P_1 and P_2, respectively (i.e., they are *redundant* for P_1 and P_2, respectively).

Now, consider:

$$W := \Big\{ \big(x^1,\, x^2,\, u\big) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^q_{\not<} :$$

$$C_1 A_1 x^1 \le C_1 a_1; \tag{8}$$

$$B_1 x^2 + B_2 x^1 - u \le 0; \tag{9}$$

$$C_2 A_2 x^2 \le C_2 a_2 \Big\} \tag{10}$$

(where $C_1 \in \mathbb{R}^{p_1 \times p_1}$ and $C_2 \in \mathbb{R}^{p_2 \times p_2}$ are diagonal matrices with positive diagonal entries).

Clearly, W *augments* P_1 and P_2 respectively. Hence:

$$W \text{ is equivalent to } P_1, \text{ and} \tag{11}$$

$$W \text{ is equivalent to } P_2. \tag{12}$$

Also clearly, we have:

$$\varphi_{x^1}(W) = P_1 \text{ [since } P_2 \neq \varnothing, \text{ and ((8) and (9) are redundant for } P_1)\text{], and} \tag{13}$$

$$\varphi_{x^2}(W) = P_2 \text{ [since } P_1 \neq \varnothing, \text{ and ((7) and (8) are redundant for } P_2)\text{].} \tag{14}$$

It follows from the combination of (11) and (14) that P_1 is an EF of P_2.

It follows from the combination of (12) and (13) that P_2 is an EF of P_1. ■

Example 1:

Let

$$P_1 = \left\{ x \in \mathbb{R}^2_{\not<} : \ 2x_1 + x_2 \le 6 \right\};$$
$$P_2 = \left\{ w \in \mathbb{R}^3_{\not<} : \ 18w_1 - w_2 \le 23; \ 59w_1 + w_3 \le 84 \right\}.$$

For arbitrary matrices B_1, B_2, C_1, and C_2 (of appropriate dimensions, respectively); say

$$B_1 = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}, \ B_2 = \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix}, \ C_1 = [7] \text{ and } C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix};$$

P_1 and P_2 can be *augmented* into EFs of each other using $u \in \mathbb{R}^2_{\not<}$ and W:

$$W = \left\{ (x, w, u) \in \mathbb{R}^{2+3+2}_{\not<} : \ [7]\,[2 \ \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le 42; \right.$$
$$\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\left. \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 18 & -1 & 0 \\ 59 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \le \begin{bmatrix} 46 \\ 42 \end{bmatrix} \right\}.$$

$\square$

4 Illustrations using some existing models

4.1 Application to the Fiorini et al. (2011, 2012a) developments

Fiorini et al. (2012a) is a re-organised and extended version of Fiorini et al. (2011). The key extension is the addition of another alternate definition of EFs [Fiorini et al. (2012a), p.96] which is recalled in this paper as Definition 3. This new alternate definition is then used to re-arrange 'Section 5' of Fiorini et al. (2011) into 'Section 2' and 'Section 3' of Fiorini et al. (2012a). Hence, the developments in 'Section 5' of Fiorini et al. (2011) which depended on 'Theorem 4' of that paper, are 'stand-alones' (as 'Section 3') in Fiorini et al. (2012a), and 'Theorem 4' in Fiorini et al. (2011) is relabelled as 'Theorem 13' in Fiorini et al. (2012a).

Claim 1: The developments in Fiorini et al. (2011) are not valid for relating the inequality descriptions of U and X in Definitions 1 to 3 when $G = \mathbf{0}$.

Proof: Using the terminology and notation of Fiorini et al. (2011), the main results of Section 2 of Fiorini et al. (2011) are developed in terms of $Q:=\{(x, y) \in \mathbb{R}^{d+k} \mid Ex + Fy = g, y \in C\}$ and $P:=\{x \in \mathbb{R}^d \mid Ax \leq b\}$, with Q (in Fiorini et al., 2011) corresponding to U in Definitions 1 to 3, and P (in Fiorini et al., 2011) corresponding to X in Definitions 1 to 3. Hence, $G = \mathbf{0}$ in Definitions 1 to 3 corresponds to $E = \mathbf{0}$ in Fiorini et al. (2011). Hence, firstly, assume $E = \mathbf{0}$ in the expression of Q (i.e., $Q:=\{(x, y) \in \mathbb{R}^{d+k} \mid \mathbf{0}x + Fy = g, y \in C\}$). Then, secondly, consider Theorem 4 of Fiorini et al. (2011) (which is pivotal in that work). We have the following:

1 If $A \neq \mathbf{0}$ in the expression of P, then the proof of the theorem is invalid since that proof requires setting '$E:=A$' [see Fiorini et al. (2011, p.7)].

2 If $A = \mathbf{0}$, then $P:=\{x \in \mathbb{R}^d \mid \mathbf{0}x \leq b\}$. This implies that either $P = \mathbb{R}^d$ (if $b \geq \mathbf{0}$) or $P = \varnothing$ (if $b \ngeq \mathbf{0}$). Hence, P would be either unbounded or empty. Hence, there could not exist a non-empty polytope, $Conv(V)$, such that $P = Conv(V)$ [see Fiorini et al. (2011, pp.16–17), among others]. Hence, the conditions in the statement of Theorem 4 of Fiorini et al. (2011) would be *ill-defined*/impossible.

Hence, the developments in Fiorini et al. (2011) are not valid for relating U and X in Definitions 1 to 3 when $G = \mathbf{0}$ in those definitions. ■

Claim 2: The developments in Fiorini et al. (2012a) are not valid for relating the inequality descriptions of U and X in Definitions 1 to 3 when $G = \mathbf{0}$.

Proof: First, note that 'Theorem 13' of Fiorini et al. (2012a, p. 101) is the same as 'Theorem 4' of Fiorini et al. (2011). Hence, the proof of Claim 1 above is applicable to 'Theorem 13' of Fiorini et al. (2012a). Hence, the parts of the developments in Fiorini et al. (2012a) that hinge on this result [namely, from 'Section 4' onward in Fiorini et al. (2012a)] are not valid for relating U and X in Definitions 1 to 3 when $G = \mathbf{0}$.

Now consider 'Theorem 3' of Fiorini et al. (2012a, Section 3, p.99). The proof of that theorem hinges on the statement that [using the terminology and notation of Fiorini et al. (2012a) which is similar to that in Fiorini et al. (2011)]:

$$Ax \leq b \Leftrightarrow \exists y : E^{\leq}x + F^{\leq}y \leq g^{\leq},\ E^{=}x + F^{=}y \leq g^{=}. \tag{15}$$

Note that G = **0** in Definitions 1 to 3 would correspond to $E^{\leq} = E^{=} = \mathbf{0}$ in Fiorini et al. (2012a). Hence, assume $E^{\leq} = E^{=} = \mathbf{0}$ in (14) above. Then, clearly, the 'if and only if' stipulation of (14) [and therefore, Fiorini et al. (2012a)] cannot be satisfied in general, since

$$(\exists y : \mathbf{0} \cdot x + F^{\leq} y \leq g^{\leq},\ \mathbf{0} \cdot x + F^{=} y \leq g^{=}) \text{ cannot imply } (Ax \leq b) \text{ in general.}$$

Hence, Theorem 3 of Fiorini et al. (2012a) is not valid for relating U and X in Definitions 1 to 3 when G = **0**.

Hence, the developments in Fiorini et al. (2012a) are not valid for relating U and X in Definitions 1 to 3 when G = **0** in those definitions. ■

4.2 The case of the MSTP: redundancy matters when '$G = \mathbf{0}$'

The consideration of 'G = **0**' we have introduced in this paper is an important one because, as we have shown, it refines the notion of an EF by separating the case in which the notion is meaningful from the case in which the notion is degenerate and ambiguous. The degeneracy (when 'G = **0**') stems from the fact that every polytope is potentially an EF of every other polytope, as we have shown in Section 3 of this paper. The ambiguity stems from the fact that one would reach contradicting conclusions as to what is/is not an EF of a given polytope, depending on what one does with the redundant constraints and variables which are introduced. This is illustrated in the following example.

Example 2 Refer back to the numerical example in Part 3 of the Proof of Theorem 1. An example of an inequality description of X in that numerical example is:

$$\overline{X} = \left\{ x \in \mathbb{R}^3_{\not<} : x_1 - x_2 + x_3 = 6; x_1 + x_2 \geq 3; x_1 + x_3 \leq 7; x_2 + x_3 \geq 6; x_1 \leq 2 \right\}.$$

(It is easy to verify that the feasible set of $\overline{X}$ is indeed $\{(2, 1, 5)^T\}$.) Let U' denote $\overline{U}$ augmented with the constraints of the linear map, $x - Aw = 0$. Clearly U' *does* project to $\overline{X}$ under the standard definition (Definition 1), whereas $\overline{U}$ does not. Hence, the answer to the question of whether or not $\overline{U}$ is an EF of $\overline{X}$ under the standard definition depends on what we do with the redundant constraints, $x - Aw = 0$. If these constraints are added to $\overline{U}$, then $\overline{U}$ becomes U', and the answer is 'Yes.' If these constraints are left out, the answer is 'No.' Hence, the EF relation which is established between $\overline{U}$ and $\overline{X}$ under Definition 2 is ambiguous (in addition to being degenerate, as we have shown in Theorem 1). □

A well-researched case in point for the discussions above is that of the MSTP. Without the refinement brought by the distinction we make between the cases of G = **0** and $G \neq \mathbf{0}$ in Definitions 1 to 3, the case of the MSTP would mean that it is possible to *extend* an exponential-sized model into a polynomial-sized one by (simply) adding redundant variables and constraints to it (i.e., *augmenting* it), which is a clearly-unreasonable/out-of-the-question proposition. To see this, assume (as is normally done in EFs work) that the addition of redundant constraints and variables does not matter as far EFs relationships are concerned. Since the constraints of Edmonds' (1970) model are redundant for the model of Martin (1991), one could *augment* Martin's formulation with these constraints. The resulting model would still be considered a polynomial-sized one.

But note that this particular *augmentation* of Martin's model would also be an *augmentation* of Edmonds' model. Hence, the conclusion would be that Edmonds' exponential-sized model has been *augmented* into a polynomial-sized one, which is an impossibility, since one cannot reduce the number of facets of a given polytope by simply adding redundant constraints to the inequality description of that polytope. The distinction we are bringing to attention in this paper explains the paradox, as further detailed below.

Example 3: We show that Martin's polynomial-sized LP model of the MSTP is not an EF (in a non-degenerate, meaningful sense) of Edmonds's exponential LP model of the MSTP, by showing that there exists a reformulation of Martin's model which does not require the variables of Edmonds' model (which is essentially the equivalent of having $G = \mathbf{0}$ in the description of U in Definitions 1 to 3).

- Using the notation in Martin(1991), i.e.:

  $N: = \{1, \ldots, n\}$ (set of vertices);

  E: set of edges;

  $\forall S \subseteq N$, $\gamma(S)$: set of edges with both ends in S.

- Exponential-sized/'sub-tour elimination' LP formulation (Edmonds, 1970):

  (P):

$$
\left|
\begin{array}{ll}
\text{Minimise}: & \sum_{e \in E} c_e x_e \\
\text{Subject to}: & \sum_{e \in E} x_e = n-1; \\
 & \sum_{e \in \gamma(S)} x_e \leq |S|-1;\ S \subset E; \\
 & x_e \geq 0 \ \text{ for all } \ e \in E.
\end{array}
\right.
$$

- Polynomial-sized LP reformulation (Martin, 1991):

  (Q):

$$
\left|
\begin{array}{ll}
\text{Minimise}: & \sum_{e \in E} c_e x_e \\
\text{Subject to}: & \sum_{e \in E} x_e = n-1; \\
 & z_{k,i,j} + z_{k,j,i} = x_e;\ k = 1, \ldots, n;\ e \in \gamma(\{i, j\}); \\
 & \sum_{s>i} z_{k,i,s} + \sum_{h<i} z_{k,i,h} \leq 1;\ \ i, k = 1, \ldots, n;\ i \neq k; \\
 & \sum_{s>k} z_{k,k,s} + \sum_{h<k} z_{k,k,h} \leq 0;\ \ k = 1, \ldots, n; \\
 & x_e \geq 0 \ \text{ for all } \ e \in E;\ \ z_{k,i,j} \geq 0 \text{ for all } k,\ i,\ j.
\end{array}
\right.
$$

- Re-statement of Martin's LP model:

  For each $e \in E$:

  1 denote the ends of e as i_e and j_e, respectively

  2 fix an arbitrary node, r_e, which is not incident on e (i.e., $r_e \notin \{i_e, j_e\}$).

- Then, one can verify that Q is equivalent to:

  (Q′):

$$\left|\begin{array}{ll} \text{Minimise:} & \sum_{e\in E} c_e z_{r_e,i_e,j_e} + \sum_{e\in E} c_e z_{r_e,j_e,i_e} \\ \text{Subject to:} & \sum_{e\in E} z_{r_e,i_e,j_e} + \sum_{e\in E} z_{r_e,j_e,i_e} = n-1; \\ & z_{k,i_e,j_e} + z_{k,j_e,i_e} = z_{r_e,i_e,j_e} + z_{r_e,j_e,i_e};\ k = 1, \ldots, n;\ e \in E; \\ & \sum_{s>i} z_{k,i,s} + \sum_{h<i} z_{k,i,h} \le 1;\quad i, k = 1, \ldots, n;\ i \neq k; \\ & \sum_{s>k} z_{k,k,s} + \sum_{h<k} z_{k,k,h} \le 0;\quad k = 1, \ldots, n; \\ & z_{k,i,j} \ge 0 \text{ for all } k,\ i,\ j. \end{array}\right.$$

□

Claim 3: We claim that the reason EFs work relating formulation sizes does not apply to the case of the MSTP is that although Martin's model can be made to *project* to Edmond's model, that projection is degenerate/non-meaningful in the sense we have described in this paper.

5 Alternate/auxiliary models

In this section, we provide some insights into the meaning of the existence of an affine map establishing a one-to-one correspondence between polytopes when the sets of their respective descriptive variables are disjoint, as brought to our attention in private e-mail communications by Kaibel (2013) and Yannakakis (2013), respectively. The linear map stipulated in Definition 2 is a special case of an affine map. Referring back to Definitions 1 to 3, we will show in this section that when $G = \mathbf{0}$ in the expression of U and there exists a one-to-one affine mapping of X onto U, then U is simply an alternate model (a 'reformulation') of X which can be used, in an 'auxiliary' way, in order to optimise any linear function of x over X, without any reference to/knowledge of an inequality description of X.

Example 4:

- Let:

  $x \in \mathbb{R}^p$ and $w \in \mathbb{R}^q$ be disjoint vectors of variables;

  $X: = \{x \in \mathbb{R}^p: Ax \le a\}$ (where $A \in \mathbb{R}^{m\times p}$, and $a \in \mathbb{R}^m$);

$U := \{w \in \mathbb{R}^q : Dw \le d\}$ (where $D \in \mathbb{R}^{n \times q}$, and $d \in \mathbb{R}^n$);
$L := \{(x, w) \in \mathbb{R}^{p+q} : x - Cw = b\}$ (where $C \in \mathbb{R}^{p \times q}$, and $b \in \mathbb{R}^p$).

- Assume that the non-negativity requirements for x and w are included in the constraints of X and U, respectively, and that L is redundant for X and for U.
- Then, it is easy to see that the optimisation problem, *Problem* LP_1:

$$\left|\begin{array}{l}\text{Minimise}: \quad \alpha^T x \\ \text{Subject to}: \; (x, w) \in L; \quad w \in U \\ (\text{where } \alpha \in \mathbb{R}^p).\end{array}\right.$$

is equivalent to the smaller linear program, *Problem* LP_2:

$$\left|\begin{array}{l}\text{Minimise}: \quad \left(\alpha^T C\right) w + \alpha^T b \\ \text{Subject to}: \; w \in U \\ (\text{where } \alpha \in \mathbb{R}^p).\end{array}\right.$$

- Hence, if L is the graph of a one-to-one correspondence between the points of X and the points of U [see Beachy and Blair (2006, pp. 47-59)], then, the optimisation of any linear function of x over X can be done by first using *Problem* LP_2 in order to get an optimal w, and then using Graph L to 'retrieve' the corresponding x. Note that the second term of the objective function of *Problem* LP_2 can be ignored in the optimisation process of *Problem* LP_2, since that term is a constant.

  Hence, if L is derived from knowledge of the vertex description of X only, then this would mean that the inequality description of X is not involved in the 'two-step' solution process (of using *Problem* LP_2 and then Graph L), but rather, that only the vertex description of X is involved. □

Hence, when $G = \mathbf{0}$, the existence of the linear map, π, stipulated in Definition 2 does not imply that U is an EF of X, but rather that U can be used to solve the optimisation problem over X without any reference to/knowledge of an inequality description of X, if π is not derived from an inequality description of X.

6 Conclusions

We have shown that EFs theory aimed at comparing and/or bounding sizes of inequality descriptions of polytopes are not applicable when the set of the descriptive variables for those polytopes are disjoint (i.e., when '$G = \mathbf{0}$'). We have illustrated our ideas using the Fiorini et al. (2011, 2012a) developments, and Martin's (1991) LP formulation of the MSTP, respectively. We have also shown that the '$G = \mathbf{0}$' consideration we have brought to attention explains the existing paradox in EFs theory (typified by the case of the MSTP), which is that by simply adding redundant constraints and variables to a model of exponential size one can obtain a model of polynomial size. A 'practical' implication of our developments is that in the context of using EFs theory to decide on the validity of a model one must show, as a first step, that there exists no (equivalent) reformulation of the model which does not require the natural variables for the optimisation problem at hand. We believe these constitute important contributions.

With respect to future research, we believe that it may be possible to delineate the scope of applicability of EFs theory further. In particular, an important question is: 'Can there be constraints of an EF of a model which are non-redundant for the model?' We believe an answer to this question could be very helpful with respect to the purpose of the further delineation of the scope of applicability of EFs theory. We are currently exploring issues such as this.

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